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Saturation Theorems for Families of Dual Operators

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1. INTRODUCTION

This paper, which continues the authors' work on the dual versions of fundamental approximation theorems [8], but can be read independently, deals with the saturation behaviour of a family of dual operators $\{T_t'; t \in (0, 1]\}$, where $\{T_t\}$ is a commutative, strong approximation process on a Banach space X satisfying a so called Voronovskaja-type relation (see (1.4) below).

Since there exist several definitions of the saturation property differing somewhat from each other (e.g. [1, p. 25; 2, p. 87; 4, p. 434; 5, p. 49], let us just recall that one which seems to be the most appropriate for our setting.

DEFINITION 1. Let $\{T_t; t \in (0, 1]\}$ be a family of bounded linear operators mapping a Banach space X into itself. $\{T_t\}$ is said to possess the saturation property, if there exists a positive function φ, defined on $(0, 1]$ with $\lim_{t \to 0+} \varphi(t) = 0$, such that: (i) for every $f \in X$ satisfying

$$\liminf_{t \to 0+} \|(\varphi(t))^{-1}[T_t f - f]\|_X = 0 \tag{1.1}$$

there holds $T_t f = f$ for small t, i.e., f is an invariant element of T_t, and (ii) the set

$$F[X; T_t] := \{f \in X; \|T_t f - f\|_X = \mathcal{O}(\varphi(t)), t \to 0+\}$$

contains at least one non-invariant element f_0.

In this event, the family $\{T_t\}$ is said to be saturated in X with order $\mathcal{O}(\varphi(t))$, and $F[X; T_t]$ is called its Favard or saturation class.

It is also possible to consider sequences of operators $\{T_k; k \in \mathbb{N}\}$ ($\mathbb{N}$ = naturals). One need just replace $t \in (0, 1]$ by $k \in \mathbb{N}$, $t \to 0+$ by $k \to \infty$.

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and $\varphi(t)$ by $\varphi(1/k)$ whenever they occur. Then all results given below will remain valid for this case.

One of the main results concerning saturation, due to H. Berens [1, Satz 3.2], is given in

THEOREM 1. *Let $\{T_t; t \in (0, 1]\}$ be a commutative, strong approximation process on a Banach space X, i.e.,*

$$T_t T_s f = T_s T_t f \qquad (f \in X; s, t \in (0, 1]), \tag{1.2}$$

$$\lim_{t \to 0+} \|T_t f - f\|_X = 0 \qquad (f \in X), \tag{1.3}$$

and B a closed linear operator with domain $D(B) \subset X$ and range $R(B) \subset X$, satisfying the Voronovskaja-type relation

$$\lim_{t \to 0+} \|(\varphi(t))^{-1}[T_t g - g] - Bg\|_X = 0 \qquad (g \in D(B)), \tag{1.4}$$

φ given as in Definition 1. *Suppose that there exists a regularization process $\{J_n; n \in \mathbb{N}\}$, i.e., a sequence of bounded linear operators from X into itself such that $J_n(X) \subset D(B)$ for each $n \in \mathbb{N}$, and*

$$\lim_{n \to \infty} \|J_n f - f\|_X = 0 \qquad (f \in X), \tag{1.5}$$

$$J_n T_t f = T_t J_n f \qquad (f \in X; n \in \mathbb{N}; t \in (0, 1]). \tag{1.6}$$

(a) *If $f \in X$ is such that* (1.1) *holds, then $f \in D(B)$ and $Bf = 0$.*

(b) *The following statements are equivalent for $f \in X$:*

(i) $\|T_t f - f\|_X = \mathcal{O}(\varphi(t))$ $(t \to 0+)$,

(ii) *$f \in \widetilde{D(B)}^X$, i.e., f belongs to the completion of $D(B)$ relative to X.*

For the definition of the relative completion recall [1, p. 14; 4, p. 373]. In (ii) as well as in the following $D(B)$ is endowed with the norm

$$\|g\|_{D(B)} := \|g\|_X + \|Bg\|_X \qquad (g \in D(B)). \tag{1.7}$$

Note that Theorem 1 does not state that $\{T_t\}$ is saturated, since the conclusion of part (a), namely, $Bf = 0$, does not necessarily imply $T_t f = f$ for small t. In many applications, however, this will be the case, so that this result is a useful tool for proving saturation theorems for particular approximation processes.

The aim of this paper now is to prove two counterparts of Theorem 1 for the family $\{T'_t\}$ of dual operators. Our results generalize in particular those of de Leeuw [7] and Butzer–Berens [2, Corollary 2.1.5] concerning saturation of dual semigroups of operators.

2. Preliminaries

Concerning notations, if X, Y are normed linear spaces, then $[X, Y]$ is the space of all bounded linear operators from X into Y, endowed with the operator norm $\|\cdot\|_{[X,Y]}$. Instead of $[X, X]$ we write $[X]$.

If X', Y' are the dual spaces of X and Y, respectively, and $T \in [X, Y]$, then the dual operator T', defined by

$$\langle T'f', f\rangle = \langle f', Tf\rangle \qquad (f' \in Y'; f \in X), \tag{2.1}$$

is an element of $[Y', X']$ and

$$\|T'\|_{[Y',X']} = \|T\|_{[X,Y]}. \tag{2.2}$$

Moreover, if Y is a Banach space and the range of T equals Y, in notation $R(T) = Y$, then T' has a continuous inverse, i.e., $(T')^{-1} \in [X', Y']$ (see [9, Section 4.5 and Theorem 4.7B]).

Now let B be a linear operator (not necessarily bounded) with domain $D(B)$ dense in X into X. The operator B^*, also called the dual of B, is a mapping whose domain $D(B^*)$ consists of all $f' \in X'$ for which there exists a $g' \in X'$ such that

$$\langle g', g\rangle = \langle f', Bg\rangle \qquad (g \in D(B)); \tag{2.3}$$

in this case one sets $B^*f' = g'$. It is clear that $D(B^*)$ is a linear manifold in X', and that B^* is a linear operator from $D(B^*)$ into X'.

On the other hand, since B becomes a bounded operator when regarded as a mapping from $D(B)$ (endowed with the norm (1.7)) into X, one can also consider the operator $B' \in [X', D(B)']$. It follows that B^* is the restriction of B' to those $f' \in X'$ for which $B'f'$ has a continuous extension from $D(B)$ to X, in other words, f' belongs to $D(B^*)$ if and only if $B'f' \in X'$. Note that the extension of $B'f'$ from $D(B)$ to X is unique since $D(B)$ is dense in X (cf. [6, p. 50]).

The following lemmas will be needed below:

Lemma 1. *Under the assumptions of Theorem* 1 *there holds*:

(i) *$D(B)$ is dense in X*;

(ii) *$T_t \in [D(B)]$ $(t \in (0, 1])$*;

(iii) *$T_tBg = BT_tg$ $(g \in D(B); t \in (0, 1])$.*

Proof. Assertion (i) follows immediately from (1.5) since $J_n f \in D(B)$. To prove (ii) and (iii) note that g belongs to $D(B)$ and $Bg = f$ if and only if $s\text{-lim}_{t\to 0+}(\varphi(t))^{-1}[T_t g - g] = f$ (cf. [4, p. 502]). Now, if $g \in D(B)$, then

$$s\text{-}\lim_{s\to 0+} \frac{T_s T_t g - T_t g}{\varphi(s)} = T_t \left[s\text{-}\lim_{s\to 0+} \frac{T_s g - g}{\varphi(s)} \right] = T_t Bg,$$

implying $T_t g \in D(B)$ and $BT_t g = T_t Bg$, which is (iii). Furthermore, part (ii) follows in view of

$$\| T_t g \|_{D(B)} = \| T_t g \|_X + \| T_t Bg \|_X \leqslant \| T_t \|_{[X]} \| g \|_{D(B)}. \quad \blacksquare$$

LEMMA 2. *Let $\{T_t; t \in (0, 1]\}$ and B be given as in Theorem* 1. *Then there exist M, $\delta > 0$ such that*

$$\|(\varphi(t))^{-1}[T_t - I]\|_{[D(B),X]} \leqslant M \qquad (0 < t \leqslant \delta), \tag{2.4}$$

$$\|(\varphi(t))^{-1}[T'_t - I]\|_{[X',D(B)']} \leqslant M \qquad (0 < t \leqslant \delta), \tag{2.5}$$

where I denotes the identity operator on any space.

The proof of (2.4) follows by the uniform boundedness principle in view of (1.4), noting that $D(B)$ is a Banach space under the norm (1.7); inequality (2.5) can be deduced from (2.4) by (2.2).

3. TWO GENERAL DUAL SATURATION THEOREMS

The aim of this section is to prove two counterparts of Theorem 1 for the family $\{T'_t; t \in (0, 1]\}$ of dual operators, where $\{T_t\}$ is given as in Theorem 1. It follows obviously from the definition and (1.3) that $\{T'_t\}$ is a family of bounded linear operators from X' into itself, satisfying

$$\lim_{t\to 0+} \langle T'_t f', f \rangle = \langle f', f \rangle \qquad (f' \in X'; f \in X),$$

i.e., T'_t tends in the w^*-topology of X' towards the identity.

Similarly, since B is a bounded linear operator from $D(B)$ normed by (1.7) into X, it follows that $B' \in [X', D(B)']$, and in view of (1.4) there holds the Voronovskaja-type relation

$$\lim_{t\to 0+} \langle (\varphi(t))^{-1}[T'_t f' - f'], g \rangle = \langle B'f', g \rangle \qquad (f' \in X'; g \in D(B)). \tag{3.2}$$

THEOREM 2. *Let $\{T_t; t \in (0, 1]\}$, B and $\{J_n; n \in \mathbb{N}\}$ be given as in Theorem* 1. *Suppose, in addition, that there exists a complex number λ such that $R(B - \lambda I) = X$.*

(a) *If* $f' \in X'$ *is such that* $\liminf_{t\to 0+} \|(\varphi(t))^{-1}[T'_t f' - f']\|_{X'} = 0$, *then* $f' \in D(B^*)$ *and* $B^* f' = 0$ *on* X.

(b) *The following statements are equivalent for* $f' \in X'$:

(i) $\|T'_t f' - f'\|_{X'} = \mathcal{O}(\varphi(t))$ $(t \to 0+)$,

(ii) $f' \in D(B^*)$.

Proof. Concerning part (a), there exists a sequence $\{t_j; j \in \mathbb{N}\} \subset (0, 1]$ with $\lim_{j\to\infty} t_j = 0$ such that

$$\lim_{j\to\infty} \|(\varphi(t_j))^{-1}[T'_{t_j} f' - f']\|_{X'} = 0.$$

This yields for all $g \in D(B)$

$$\begin{aligned} 0 &= \lim_{j\to\infty} \langle (\varphi(t_j))^{-1}[T'_{t_j} f' - f'], g\rangle \\ &= \lim_{j\to\infty} \langle f', (\varphi(t_j))^{-1}[T_{t_j} g - g]\rangle = \langle f', Bg\rangle, \end{aligned}$$

giving $f' \in D(B^*)$ and $B^* f' = 0$ by (2.3).

As to part (b), setting $B_t := (\varphi(t))^{-1}[T_t - I]$, one has by (i) that the family $\{B'_t f'\} \subset X'$, regarded as a family of operators from X into the space of complex numbers, is uniformly bounded for $0 < t \leqslant \delta$, and strongly convergent for $t \to 0+$ on a dense subset of X by (3.2). An application of the Banach–Steinhaus theorem (cf. [4, Proposition 0.7.3]) then yields the existence of a $g' \in X'$ such that

$$\lim_{t\to 0+} \langle B'_t f', f\rangle = \langle g', f\rangle \qquad (f \in X).$$

Comparing this with (3.2) reveals that

$$\langle g', g\rangle = \langle B' f', g\rangle = \langle f', Bg\rangle \qquad (g \in D(B)),$$

and so it follows that $f' \in D(B^*)$.

Conversely, since $R(B - \lambda I) = X$ implies that $B' - \lambda I$ has a continuous inverse from $D(B)'$ into X', one obtains

$$\begin{aligned} \|B'_t f' - B^* f'\|_{X'} &= \|(B' - \lambda I)^{-1}(B' - \lambda I)(B'_t - B^*) f'\|_{X'} \\ &\leqslant M \|(B' - \lambda I)(B'_t - B^*) f'\|_{D(B)'} \\ &= M \|(B' - \lambda I)(B'_t - B') f'\|_{D(B)'}, \end{aligned}$$

noting that $B' f' = B^* f'$ on $D(B)$. Now one easily verifies by Lemma 1(iii) that $(B' - \lambda I)$ commutes with $(B_t' - B')$, and so in view of (2.5) there follows the estimate

$$\|B_t'f' - B^*f'\|_{X'} \leqslant M \|B_t' - B'\|_{[X', D(B)']} \|(B' - \lambda I)f'\|_{X'} = \mathcal{O}(1) \qquad (t \to 0+).$$

This gives assertion (i), and the proof is complete. ▮

It should be mentioned that the existence of the regularization process $\{J_n\}$, which is not explicitely used in the proof of Theorem 2, is needed for the proof of Lemma 1(iii) which in turn was used to show that $(B' - \lambda I)$ commutes with $(B_t' - B')$. If T_t maps X into $D(B)$, which is the case in many applications, then one may take $J_n = T_{1/n}$.

Now we come to the second counterpart of Theorem 1. Since T_t belongs also to $[D(B)]$, one can in addition treat saturation of $\{T_t'\}$ in $[D(B)']$. In this case one has to assume that the J_n map X continuously into $D(B)$.

LEMMA 3. *Let $\{T_t\}$, B and $\{J_n\}$ be given as in Theorem* 1, *and assume that $J_n \in [X, D(B)]$ for each $n \in \mathbb{N}$. Then*

(i) $J_n Bg = BJ_n g \ (g \in D(B); n \in \mathbb{N})$;

(ii) $\lim_{n\to\infty} \|J_n g - g\|_{D(B)} = 0 \ (g \in D(B))$;

(iii) $\|J_n\|_{[D(B)]} \leqslant M \ (n \in \mathbb{N})$;

(iv) $J_n' B' g' = B' J_n' g' \ (g' \in D(B)'; n \in \mathbb{N})$;

(v) $\lim_{n\to\infty} \langle J_n' f', g\rangle = \langle f', g\rangle \ (f' \in D(B)'; g \in D(B))$;

(vi) $\|J_n'\|_{[D(B)']} \leqslant M \ (n \in \mathbb{N})$.

Proof. Assertion (i) can be proved similarly as Lemma 1(iii) using (1.6); (ii) then follows by (1.5). Regarding (iii), one has again to apply the Banach–Steinhaus theorem, noting (ii) and the fact that $\{J_n\} \subset [D(B)]$. Statements (iv), (v) and (vi), finally, are the dual versions of (i), (ii) and (iii). ▮

Our second result now reads

THEOREM 3. *Let the assumptions of Theorem* 2 *be satisfied and suppose, in addition, that $\{J_n; n \in \mathbb{N}\} \subset [X, D(B)]$.*

(a) *If $f' \in D(B)'$ is such that* $\liminf_{t\to 0+} \|(\varphi(t))^{-1}[T_t'f' - f']\|_{D(B)'} = 0$, *then f' has a continuous extension from $D(B)$ to X, belongs to $D(B^*)$ and $B^*f' = 0$ on X.*

(b) *The following statements are equivalent for $f' \in D(B)'$:*

(i) $\|T_t'f' - f'\|_{D(B)'} = \mathcal{O}(\varphi(t)) \ (t \to 0+)$,

(ii) *f' has a continuous extension from $D(B)$ to X, i.e., $f' \in X'$.*

Proof. We prove only part (b) since (a) follows by the same argument as in the proof of Theorem 2. Now for $f' \in D(B)'$ one has $J'_n f' \in X'$, and

$$\|J'_n f'\|_{X'} = \|(B' - \lambda I)^{-1}(B' - \lambda I) J'_n f'\|_{X'}$$
$$\leqslant M\{\|B' J'_n f'\|_{D(B)'} + |\lambda| \, \|J'_n f'\|_{D(B)'}\}.$$

Using (1.6) and (i) one can estimate the first term by

$$\|B' J'_n f'\|_{D(B)'} \leqslant \liminf_{t \to 0+} \|J'_n[(\varphi(t))^{-1}[T'_t f' - f']]\|_{D(B)'}$$
$$\leqslant \sup_{t \in (0,\delta)} \|(\varphi(t))^{-1}[T'_t f' - f']\|_{D(B)'} \|J'_n\|_{[D(B)']}$$
$$\leqslant M \|J'_n\|_{[D(B)']} \qquad (n \in \mathbb{N}),$$

so that together

$$\|J'_n f'\|_{X'} \leqslant \{M + |\lambda| \, \|f'\|_{D(B)'}\} \, \|J'_n\|_{[D(B)']} \leqslant M_1 \qquad (n \in \mathbb{N}),$$

the latter inequality being valid by Lemma 3(vi). So the sequence $\{J'_n f'\}$ is uniformly bounded with respect to $n \in \mathbb{N}$, as well as convergent on a dense subset of X by Lemma 3(v). So one can conclude that there exists some $f'_0 \in X'$ satisfying

$$\lim_{n \to \infty} \langle J'_n f', f \rangle = \langle f'_0, f \rangle \qquad (f \in X).$$

Comparing this result with Lemma 3(v) shows that f'_0 is the desired extension of f'. The converse direction, finally, is given by inequality (2.5). ∎

The difference between Theorems 2 and 3 is that they treat the saturation problem in different spaces. In Theorem 2 the operators T'_t are regarded as elements of $[X']$, whereas in Theorem 3 they are considered as elements of $[D(B)']$. Although the proofs of both theorems are quite similar, it does not seem that one can be deduced from the other.

4. Applications

4.1. *Semigroups of Operators*

If $\{S(t); t \geqslant 0\}$ is a (C_0)-semigroup of operators defined on a Banach space X (for definition see [2, Section 1.1]), and A is the infinitesimal generator, then $D(A)$ is dense in X, and by definition there holds

$$\lim_{t \to 0+} \|t^{-1}[S(t) g - g] - Ag\|_X = 0 \qquad (g \in D(A)),$$

i.e., $\{S(t); t \geqslant 0\}$ satisfies a Voronovskaja-type relation with $\varphi(t) = t$ and $B = A$. Moreover,

$$J_n f := n \int_0^{1/n} S(t) f \, dt \qquad (n \in \mathbb{N})$$

defines a regularization process, as needed in Theorems 1 or 2, and one has that $R(A - \lambda I) = X$ for all complex λ with real part large enough (cf. [2, pp. 31, 32]). Finally, if $f' \in D(A^*)$, then (cf. [2, p. 48])

$$\langle S(t)' f' - f', f \rangle = \int_0^t \langle S(u)' A^* f', f \rangle \, du \qquad (f \in X; t > 0).$$

This shows that $A^* f' = 0$ on X implies $S(t)' f' = f'$ for all $t > 0$.

As an application of Theorem 2 one now obtains

THEOREM 4. *Let $\{S(t); t \geqslant 0\}$ be a (C_0)-semigroup of operators on a Banach space X with infinitesimal generator A:*

(a) *If $f' \in X'$ is such that* $\liminf_{t \to 0+} \| t^{-1} [S(t)' f' - f'] \|_{X'} = 0$, *then $S(t)' f' = f'$ for all $t > 0$.*

(b) *The following statements are equivalent for $f' \in X'$:*

(i) $\| S(t)' f' - f' \|_{X'} = \mathcal{O}(t)$ $(t \to 0+)$,

(ii) $f' \in D(A^*)$.

Theorem 4 states that $\{S(t)'\}$ is saturated in X' with order $\mathcal{O}(t)$, the saturation class being given by $D(A^*)$. Note that this result was already established by de Leeuw [7] using the Banach–Alaoglu theorem on w^*-compactness of bounded w^*-closed sets; see also [2, Theorem 2.1.4].

In order to apply Theorem 2, we restrict ourselves to holomorphic semigroups, which means that $S(t) f \in D(A)$ for all $f \in X$ and $t > 0$. In this case one can use the results in [8] to obtain in addition assertions on non-optimal approximation.

THEOREM 5. *Let $\{S(t); t \geqslant 0\}$ be a holomorphic (C_0)-semigroup of operators on a Banach space X.*

(a) *The following assertions are equivalent for $f' \in D(A)'$ and $0 < \sigma < 1$:*

(i) $\| S(t)' f' - f' \|_{D(A)'} = \mathcal{O}(t^\sigma)$ $(t \to 0+)$,

(ii) $K(t^\sigma, f'; D(A)', X') = \mathcal{O}(t^\sigma)$ $(t \to 0+)$.

(b) *$\{S(t)'\}$ is saturated in $D(A)'$ with order $\mathcal{O}(t)$, the saturation class is given by X'.*

For the proof of part (a) and the definition of the K-functional see [8]. The Jackson-type inequality needed there follows from (2.4) and the Bernstein-type inequality from [2, Proposition 1.1.11]. Note that Theorem 5(b) remains valid if the assumption $\{S(t)\}$ to be holomorphic is dropped.

4.2. *Convolution Integrals*

As a further application of Theorem 3 we consider approximation processes generated by convolution integrals. Let $C_{2\pi}$ denote the space of all continuous, 2π-periodic, complex-valued functions defined on the real axis $\mathbb{R}$, endowed with the supremum norm $\|f\|_C$. A sequence of functions $\{\chi_k: k \in \mathbb{N}\}$ in $C_{2\pi}$ is called an approximate identity, if $\int_{-\pi}^{\pi} \chi_k(u)\,du = 2\pi$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \int_{|u|\geqslant\delta} |\chi_k(u)|\,du = 0$ for each $\delta > 0$. The convolution integrals of $f \in C_{2\pi}$ with χ_k are defined as

$$(V_k f)(x) \equiv (f * \chi_k)(x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u)\,\chi_k(u)\,du \qquad (k \in \mathbb{N};\, x \in \mathbb{R}).$$

The V_k are bounded linear operators from $C_{2\pi}$ into itself satisfying

$$V_k V_j f = V_j V_k f, \qquad \lim_{k\to\infty} \|V_k f - f\|_C = 0 \qquad (f \in C_{2\pi};\, j, k \in \mathbb{N}).$$

In order to apply Theorem 2 we assume that the V_k satisfy a Voronovskaja-type relation with respect to the 2nd derivative, namely,

$$\lim_{k\to\infty} \|k^\alpha [V_k g - g] - c g^{(2)}\|_C = 0 \qquad (g \in C_{2\pi}^2) \tag{4.1}$$

for some $\alpha > 0$, $c \in \mathbb{R}\backslash\{0\}$, where $C_{2\pi}^2$ is the set of all $g \in C_{2\pi}$ for which the 2nd derivative $g^{(2)}$ again belongs to $C_{2\pi}$. So we have $B = c(d/dx)^2$, $D(B) = C_{2\pi}^2$ and $\varphi(1/k) = k^{-\alpha}$.

As regularization operators J_n one can take any convolution integral with underlying approximate identity $\{\kappa_n\} \subset C_{2\pi}^2$. Finally, one has that for each complex λ with $\lambda/c \neq j^2$ for $j = 0, 1, 2,...$ there holds $R(c(d/dx)^2 - \lambda I) = C_{2\pi}$.

In order to compute the dual space of $C_{2\pi}^2$ we regard $C_{2\pi}'$ and $(C_{2\pi}^2)'$ as subspaces of $\mathscr{D}_{2\pi}'$, the space of all 2π-periodic distributions (cf. [10, Chapter 11]). On $\mathscr{D}_{2\pi}'$ we consider the operator

$$(I^2 f')(x) := \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} (ij)^{-2} f'^{\wedge}(j)\, e^{ijx} \qquad (f' \in \mathscr{D}_{2\pi}'),$$

where convergence is to be understood in the topology of $\mathscr{D}_{2\pi}'$, and the distributional Fourier coefficients are given by $f'^{\wedge}(j) := (2\pi)^{-1}\langle f'(x), e^{-ijx}\rangle$.

Setting now

$$(C'_{2\pi})^{-2} := \{f' \in \mathscr{D}'_{2\pi}; I^2 f' \in C'_{2\pi}\},$$
$$\|f'\|_{(C'_{2\pi})^{-2}} := |f'^{\wedge}(0)| + \|I^2 f'\|_{C'_{2\pi}},$$

then $(C^2_{2\pi})'$ and $(C'_{2\pi})^{-2}$ are equal with equivalent norms (cf. [8]).

THEOREM 6. *Let* $\{V_k; k \in \mathbb{N}\}$ *be a sequence of convolution integrals satisfying* (4.1) *for some* $\alpha > 0$, $c \in \mathbb{R}\backslash\{0\}$.

(a) *If* $f' \in (C'_{2\pi})^{-2}$ *is such that* $\liminf_{k\to\infty} \|k^\alpha [V'_k f' - f']\|_{(C'_{2\pi})^{-2}} = 0$, *then* $V'_k f' = f'$ for all $k \in \mathbb{N}$.

(b) *The following statements are equivalent for* $f' \in (C'_{2\pi})^{-2}$:

(i) $\|V'_k f' - f'\|_{(C'_{2\pi})^{-2}} = \mathcal{O}(k^{-\alpha})$ $(k \to \infty)$,

(ii) $f' \in C'_{2\pi}$.

There are various convolution integrals satisfying the Voronovskaja-type relation (4.1), e.g., the integrals of Jackson with $\alpha = 2$, $c = 3/2$, of de La Vallée Poussin with $\alpha = 1$, $c = 1$ and those of Rogosinski with $\alpha = 2$, $c = \pi^2/8$ (cf. [3]).

Similarly as in Theorem 5 one can use the results in [8] to treat approximation orders $\mathcal{O}(k^{-\sigma})$ for $0 < \sigma < \alpha$. Results corresponding to those of Theorem 6 are valid in $L^p_{2\pi}$-spaces.

Of course it would also be possible to apply Theorem 2 to convolution integrals. In this case one has to compute the spaces $D(B^*)$. For some particular integrals this is carried out in [7].

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